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Original Article

Some constraints of hypergeometric functions to belong to certain subclasses of analytic functions

M.K. Aouf [a], A.O. Mostafa [a], H.M. Zayed [b,*]

[a] Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
[b] Department of Mathematics, Faculty of Science, Menofia University, Shebin Elkom 32511, Egypt

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Abstract The purpose of this paper is to introduce sufficient conditions for (Gaussian) hypergeometric functions to be in various subclasses of analytic functions. Also, we investigate several mapping properties involving these subclasses.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic and univalent in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. For $g(z) \in \mathcal{A}$ of the form:

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n, \tag{2}$$

the Hadamard product (or convolution) of two power series $f(z)$ and $g(z)$ is given by (see [1]):

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n g_n z^n = (g * f)(z). \tag{3}$$

We recall some definitions which will be used in our paper.

Definition 1.1. For two functions $f(z)$ and $g(z)$, analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and written $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$

* Corresponding author. Tel.: +20 1090388351.
E-mail addresses: mkaouf127@yahoo.com (M.K. Aouf), adelaeg254@yahoo.com (A.O. Mostafa), hanaa_zayed42@yahoo.com (H.M. Zayed).
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$(z \in \mathbb{U})$. Furthermore, if the function $g(z)$ is univalent in $\mathbb{U}$, then we have the following equivalence (see [2]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Definition 1.2. A function $f(z) \in \mathcal{A}$ is called starlike of order α (denoted by $\mathcal{S}^*(\alpha)$), if $f(z)$ satisfies the following condition:

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \ (0 \leq \alpha < 1;\ z \in \mathbb{U}). \tag{4}$$

Also, a function $f(z) \in \mathcal{A}$ is called convex of order α (denoted by $\mathcal{K}(\alpha)$), if $f(z)$ satisfies the following condition:

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \ (0 \leq \alpha < 1;\ z \in \mathbb{U}). \tag{5}$$

The classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ were studied by MacGregor [3], Schild [4], Pinchuk [5] and others. From (4) and (5) we can see that

$$f(z) \in \mathcal{K}(\alpha) \Longleftrightarrow zf'(z) \in \mathcal{S}^*(\alpha). \tag{6}$$

We denote by $\mathcal{S}^* = \mathcal{S}^*(0)$ and $\mathcal{K} = \mathcal{K}(0)$, where $\mathcal{S}^*$ and $\mathcal{K}$ are the classes of starlike and convex functions, respectively, (see Robertson [6]).

Definition 1.3. A function $f(z) \in \mathcal{A}$ is said to be k-uniformly convex function (denoted by $k - \mathcal{UCV}$), if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq k\left|\frac{zf''(z)}{f'(z)}\right| \ (k \geq 0;\ z \in \mathbb{U}). \tag{7}$$

Also, a function $f(z) \in \mathcal{A}$ is said to be k -starlike function (denoted by $k - \mathcal{ST}$), if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq k\left|\frac{zf'(z)}{f(z)} - 1\right| \ (k \geq 0;\ z \in \mathbb{U}). \tag{8}$$

The classes of k-uniformly convex functions and k-starlike functions were introduced by Kanas and Wisniowska (see [7,8]).

Definition 1.4 [9, with $p = 1$]**.** For $-1 \leq A < B \leq 1$, $|\lambda| < \frac{\pi}{2}$ and $0 \leq \alpha < 1$, we say that a function $f(z) \in \mathcal{A}$ is in the class $R^\lambda(A, B, \alpha)$ if it satisfies the subordination condition:

$$e^{i\lambda} f'(z) \prec \cos\lambda\left[(1-\alpha)\frac{1+Az}{1+Bz} + \alpha\right] + i\sin\lambda. \tag{9}$$

Using the principle of subordination, $f(z) \in R^\lambda(A, B, \alpha)$ if and only if there exists function $w(z)$ satisfying $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$ such that

$$e^{i\lambda} f'(z) = \cos\lambda\left[(1-\alpha)\frac{1+Aw(z)}{1+Bw(z)} + \alpha\right] + i\sin\lambda,$$

or, equivalently,

$$\left|\frac{e^{i\lambda}(f'(z)-1)}{Be^{i\lambda}f'(z) - [Be^{i\lambda} + (A-B)(1-\alpha)\cos\lambda]}\right| < 1 \ (z \in \mathbb{U}). \tag{10}$$

For suitable choices of A, B and α, we obtain the following subclasses:

(i) $R^\lambda(-1, 1, \alpha) = R^\lambda(\alpha)$ $(0 \leq \alpha < 1)$ (see Kanas and Srivastava [10]);

(ii) $R^\lambda(A, B, 0) = R^\lambda(A, B)$ $\left(-1 \leq A < B \leq 1,\ |\lambda| < \frac{\pi}{2}\right)$ (see Shukla and Dashrath [11]);

(iii) $R^0(-\beta, \beta, 0) = D(\beta)$ the class of functions $f(z) \in \mathcal{A}$ satisfying the condition

$$\left|\frac{f'(z)-1}{f'(z)+1}\right| < \beta \ (0 < \beta \leq 1;\ z \in \mathbb{U}),$$

introduced and studied by Padmanabhan [12] and later Caplinger and Causey [13];

(iv) $R^0(-\beta, \beta, \alpha) = R(\alpha, \beta)$ the class of functions $f(z) \in \mathcal{A}$ satisfying the condition

$$\left|\frac{f'(z)-1}{f'(z)+1-2\alpha}\right| < \beta \ (0 \leq \alpha < 1;\ 0 < \beta \leq 1;\ z \in \mathbb{U}),$$

studied by Junenja and Mogra [14].

Also, we note that:
$R^\lambda(-\beta, \beta, \alpha) = R^\lambda(\alpha, \beta) =$

$$\left\{f(z) \in \mathcal{A} : \left|\frac{f'(z)-1}{f'(z)-1+2(1-\alpha)e^{-i\lambda}\cos\lambda}\right| < \beta \ (|\lambda| < \frac{\pi}{2};\ 0 \leq \alpha < 1;\ 0 < \beta \leq 1;\ z \in \mathbb{U})\right\}.$$

The (Gaussian) hypergeometric function ${}_2F_1(a, b; c; z)$ is defined by

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} z^n \ (z \in \mathbb{U};\ c \neq 0, -1, -2, \ldots),$$

where

$$(\gamma)_n = \begin{cases} 1 & \text{if } n = 0, \\ \gamma(\gamma+1)(\gamma+2)\ldots(\gamma+n-1) & \text{if } n \in \mathbb{N} = \{1, 2, \ldots\}. \end{cases}$$

We note that ${}_2F_1(a, b; c; 1)$ converges for $\operatorname{Re}(c - a - b) > 0$ and is related to gamma function by

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{11}$$

Using the (Gaussian) hypergeometric function, consider the functions

$$g(a, b; c; z) = z\,{}_2F_1(a, b; c; z) \ (z \in \mathbb{U}), \tag{12}$$

$$h_\mu(a, b; c; z) = (1-\mu)[g(a, b; c; z)] + \mu z[g(a, b; c; z)]' \ (z \in \mathbb{U};\ \mu \geq 0), \tag{13}$$

and

$$J_{\mu,\delta}(a, b; c; z) = (1-\mu+\delta)[g(a, b; c; z)] + (\mu-\delta)z[g(a, b; c; z)]' + \mu\delta z^2[g(a, b; c; z)]'' \ (z \in \mathbb{U};\ \mu, \delta \geq 0;\ \mu \geq \delta). \tag{14}$$

The mapping properties of functions $h_\mu(a, b; c; z)$ and $J_{\mu,\delta}(a, b; c; z)$ were studied by Shukla and Shukla [15] and Tang and Deng [16, with $p = 1$], respectively.

For a function $f(z) \in \mathcal{A}$ belonging to the class k -uniformly convex functions, denoted by $k - \mathcal{UCV}$, the following holds (see [7]):

$$|a_n| \leq \frac{(P_1)_{n-1}}{n!} \ (n \geq 2), \tag{15}$$

where $P_1 = P_1(k)$ is the coefficient of z in the function

$$p_k(z) = 1 + \sum_{n=1}^{\infty} P_n(k) z^n, \tag{16}$$

which is the extremal function for the class $\mathcal{P}(p_k)$ related to the class $k - \mathcal{UCV}$ by the range of the expression $1 + \frac{zf''(z)}{f'(z)}$ $(z \in \mathbb{U})$. Similarly, if $f(z) \in \mathcal{A}$ belongs to the class k-starlike functions denoted by $k - \mathcal{ST}$, then (see [8]):

$$|a_n| \leq \frac{(P_1)_{n-1}}{(n-1)!} \ (n \geq 2). \tag{17}$$

Using the Gaussian hypergeometric function ${}_2F_1(a, b; c; z)$, Hohlov (see [17]) defined the linear operator $I_{a,b,c} : \mathcal{A} \to \mathcal{A}$ by the convolution

$$[I_{a,b,c}(f)](z) = z {}_2F_1(a, b; c; z) * f(z) \ (f \in \mathcal{A}). \tag{18}$$

2. Main results

Unless otherwise mentioned, we assume throughout this paper that $\mathbb{C}\backslash\{0\} = \mathbb{C}^*$, $-1 \leq A < B \leq 1$, $|\lambda| < \dfrac{\pi}{2}$, $0 \leq \alpha < 1$, $k \geq 0$, $\mu, \delta \geq 0$ and $\mu \geq \delta$.

To establish our results, we need the following lemmas.

Lemma 2.1 *[9, Theorem 4, with p = 1]. A sufficient condition for f(z) defined by (1.1) to be in the class $R^{\lambda}(A, B, \alpha)$ is*

$$\sum_{n=2}^{\infty} n(1 + |B|)|a_n| \leq (B - A)(1 - \alpha)\cos\lambda. \tag{19}$$

Lemma 2.2 *[9, Theorem 1, with p = 1]. Let the function f(z) defined by (1.1) be in the class $R^{\lambda}(A, B, \alpha)$, then*

$$|a_n| \leq \frac{(B - A)(1 - \alpha)\cos\lambda}{n} \ (n \geq 2). \tag{20}$$

Lemma 2.3 *[7]. Let $f(z) \in \mathcal{A}$. If for some k, the following inequality*

$$\sum_{n=2}^{\infty} n(n-1)|a_n| \leq \frac{1}{k+2}, \tag{21}$$

holds, then $f \in k - \mathcal{UCV}$. The number $\frac{1}{k+2}$ cannot be increased.

Lemma 2.4 *[8]. Let $f(z) \in \mathcal{A}$. If for some k, the following inequality*

$$\sum_{n=2}^{\infty} [n + k(n-1)]|a_n| \leq 1, \tag{22}$$

holds, then $f \in k - \mathcal{ST}$.

Theorem 2.1. *Let $a, b \in \mathbb{C}^*$ and c be a real number such that $c > |a| + |b| + 1$. Then the sufficient condition for $g(a, b; c; z)$ to be in the class $R^{\lambda}(A, B, \alpha)$ is that*

$$\frac{\Gamma(c)\Gamma(c - |a| - |b|)}{\Gamma(c - |a|)\Gamma(c - |b|)}\left[1 + \frac{|ab|}{(c - |a| - |b| - 1)}\right] \leq \left[1 + \frac{(B - A)(1 - \alpha)\cos\lambda}{(1 + |B|)}\right]. \tag{23}$$

Proof. Since

$$g(a, b; c; z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} z^n \ (z \in \mathbb{U}),$$

According to Lemma 2.1, we need only to show that

$$\sum_{n=2}^{\infty} n(1 + |B|)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right| \leq (B - A)(1 - \alpha)\cos\lambda. \tag{24}$$

Since

$$|(d)_n| \leq (|d|)_n, \tag{25}$$

then, the left hand side of (24) is less than or equal to

$$\sum_{n=2}^{\infty} n(1 + |B|)\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} = T_1.$$

So, we obtain

$$\begin{aligned}
T_1 &= (1 + |B|)\left[\sum_{n=2}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} + \sum_{n=2}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-2}}\right] \\
&= (1 + |B|)\left[\frac{\Gamma(c)\Gamma(c - |a| - |b|)}{\Gamma(c - |a|)\Gamma(c - |b|)}\right. \\
&\quad \left. + \frac{|ab|}{c}\frac{\Gamma(c + 1)\Gamma(c - |a| - |b| - 1)}{\Gamma(c - |a|)\Gamma(c - |b|)} - 1\right] \\
&= (1 + |B|)\frac{\Gamma(c)\Gamma(c - |a| - |b|)}{\Gamma(c - |a|)\Gamma(c - |b|)}\left[1 + \frac{|ab|}{(c - |a| - |b| - 1)}\right] \\
&\quad - (1 + |B|).
\end{aligned}$$

But this last expression is bounded above by $(B - A)(1 - \alpha)\cos\lambda$ if (23) holds. This completes the proof of Theorem 2.1. $\square$

Theorem 2.2. *Let $a, b \in \mathbb{C}^*$ and c be a real number such that $c > |a| + |b| + 2$. Then the sufficient condition for $h_\mu(a, b; c; z)$ to be in the class $R^{\lambda}(A, B, \alpha)$ is that*

$$\begin{aligned}
&\frac{\Gamma(c)\Gamma(c - |a| - |b|)}{\Gamma(c - |a|)\Gamma(c - |b|)}\left[1 + (1 + 2\mu)\frac{|ab|}{(c - |a| - |b| - 1)}\right. \\
&\quad \left. + \frac{\mu(|a|)_2(|b|)_2}{(c - |a| - |b| - 2)_2}\right] \\
&\quad \leq \left[1 + \frac{(B - A)(1 - \alpha)\cos\lambda}{(1 + |B|)}\right].
\end{aligned} \tag{26}$$

Proof. Clearly $h_\mu(a, b; c; z)$ has the series representation

$$h_\mu(a, b; c; z) = z + \sum_{n=2}^{\infty} [1 + \mu(n-1)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} z^n \ (z \in \mathbb{U}),$$

by Lemma 2.1, it is enough to show that

$$\sum_{n=2}^{\infty} n(1+|B|)\left|[1+\mu(n-1)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right| \leq (B-A)(1-\alpha)\cos\lambda. \tag{27}$$

Using (25), the left hand side of (27), is less than or equal to

$$\sum_{n=2}^{\infty} n(1+|B|)[1+\mu(n-1)]\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} = T_2.$$

Now

$$\begin{aligned} T_2 &= (1+|B|)\left[\sum_{n=2}^{\infty}[1+\mu(n-1)]\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-2}} + \sum_{n=2}^{\infty}[1+\mu(n-1)]\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right] \\ &= (1+|B|)\left[\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} + (1+2\mu)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-2}} + \mu\sum_{n=3}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-3}}\right] \\ &= (1+|B|)\left[\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)} + (1+2\mu)\frac{|ab|}{c}\frac{\Gamma(c+1)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)} + \mu\frac{(|a|)_2(|b|)_2}{(c)_2}\frac{\Gamma(c+2)\Gamma(c-|a|-|b|-2)}{\Gamma(c-|a|)\Gamma(c-|b|)} - 1\right] \\ &= (1+|B|)\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[1+(1+2\mu)\times\frac{|ab|}{(c-|a|-|b|-1)} + \mu\frac{(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2}\right] - (1+|B|), \end{aligned}$$

and this last expression is bounded above by $(B-A)(1-\alpha)\cos\lambda$ if (26) holds. This ends the proof of Theorem 2.2. □

Theorem 2.3. *Let $a, b \in \mathbb{C}^*$ and c be a real number such that $c > |a|+|b|+3$. Then the sufficient condition for $J_{\mu,\delta}(a,b;c;z)$ to be in the class $R^{\lambda}(A,B,\alpha)$ is that*

$$\begin{aligned} &\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[1+(1+2\mu-2\delta+4\mu\delta)\times\frac{|ab|}{(c-|a|-|b|-1)} + (\mu-\delta+5\mu\delta)\times\frac{(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2} + \mu\delta\frac{(|a|)_3(|b|)_3}{(c-|a|-|b|-3)_3}\right] \\ &\leq \left[1+\frac{(B-A)(1-\alpha)\cos\lambda}{(1+|B|)}\right]. \end{aligned} \tag{28}$$

Proof. By means of (14), we obtain

$$J_{\mu,\delta}(a,b;c;z) = z + \sum_{n=2}^{\infty}[1+(n-1)(\mu-\delta+n\mu\delta)] \times \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}z^n \ (z\in\mathbb{U}).$$

From Lemma 2.1, we need only to prove that

$$T_3 = \sum_{n=2}^{\infty} n(1+|B|)\left|[1+(n-1)(\mu-\delta+n\mu\delta)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right| \leq (B-A)(1-\alpha)\cos\lambda. \tag{29}$$

Using (25), we have

$$\begin{aligned} T_3 &\leq \sum_{n=2}^{\infty} n(1+|B|)[1+(n-1)(\mu-\delta) + n(n-1)\mu\delta)]\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\ &= (1+|B|)\left[\sum_{n=2}^{\infty} n\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} + \sum_{n=2}^{\infty} n(n-1)(\mu-\delta)\times\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} + \sum_{n=2}^{\infty} n^2(n-1)\mu\delta\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right] \\ &= (1+|B|)\left[\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} + (1+2\mu-2\delta+4\mu\delta)\times\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-2}} + (\mu-\delta+5\mu\delta)\times\sum_{n=3}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-3}} + \mu\delta\sum_{n=4}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-4}}\right] \\ &= (1+|B|)\left[\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)} + (1+2\mu-2\delta+4\mu\delta)\frac{|ab|}{c}\frac{\Gamma(c+1)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)} + (\mu-\delta+5\mu\delta)\frac{(|a|)_2(|b|)_2}{(c)_2}\frac{\Gamma(c+2)\Gamma(c-|a|-|b|-2)}{\Gamma(c-|a|)\Gamma(c-|b|)} + \mu\delta\frac{(|a|)_3(|b|)_3}{(c)_3}\frac{\Gamma(c+3)\Gamma(c-|a|-|b|-3)}{\Gamma(c-|a|)\Gamma(c-|b|)} - 1\right] \\ &= (1+|B|)\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[1+(1+2\mu-2\delta+4\mu\delta)\times\frac{|ab|}{(c-|a|-|b|-1)} + (\mu-\delta+5\mu\delta)\frac{(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2} + \mu\delta\frac{(|a|)_3(|b|)_3}{(c-|a|-|b|-3)_3}\right] - (1+|B|). \end{aligned}$$

The proof now follows by (29). □

Theorem 2.4. *Let $a, b \in \mathbb{C}^*$ and c be a real number such that $c > |a|+|b|+2$. If the following inequality*

$$\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[1+\frac{3|ab|}{(c-|a|-|b|-1)} + \frac{(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2}\right] \leq \left[1+\frac{(B-A)(1-\alpha)\cos\lambda}{1+|B|}\right], \tag{30}$$

holds, then $[I_{a,b,c}(f)](z)$ maps the class $\mathcal{S}^*$ to the class $R^{\lambda}(A,B,\alpha)$.

Proof. Since

$$[I_{a,b,c}(f)](z) = z{}_2F_1(a,b;c;z) * f(z) = z + \sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n z^n \ (z\in\mathbb{U}).$$

It suffices to show that

$$T_4=\sum_{n=2}^{\infty} n(1+|B|)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right||a_n| \le (B-A)(1-\alpha)\cos\lambda. \tag{31}$$

By making use of (25) and the fact that $f(z)\in\mathcal{S}^*$ (i.e., $|a_n|\le n$ if $f(z)\in\mathcal{A}$) (see [1]), we get

$$\begin{aligned}
T_4 &\le \sum_{n=2}^{\infty} n^2(1+|B|)\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}\\
&= (1+|B|)\left[\sum_{n=3}^{\infty} n\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-2}}+\sum_{n=2}^{\infty} n\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right]\\
&= (1+|B|)\left[\sum_{n=3}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-3}}+3\sum_{n=2}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-2}}\right.\\
&\quad\left.+\sum_{n=2}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right]\\
&= (1+|B|)\left[\frac{(|a|)_2(|b|)_2}{(c)_2}\sum_{n=0}^{\infty}\frac{(|a|+2)_n(|b|+2)_n}{(c+2)_n(1)_n}\right.\\
&\quad\left.+3\frac{|ab|}{c}\sum_{n=0}^{\infty}\frac{(|a|+1)_n(|b|+1)_n}{(c+1)_n(1)_n}+\sum_{n=1}^{\infty}\frac{(|a|)_n(|b|)_n}{(c)_n(1)_n}\right]\\
&= (1+|B|)\left[\frac{(|a|)_2(|b|)_2}{(c)_2}\frac{\Gamma(c+2)\Gamma(c-|a|-|b|-2)}{\Gamma(c-|a|)\Gamma(c-|b|)}\right.\\
&\quad+3\frac{|ab|}{c}\frac{\Gamma(c+1)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)}\\
&\quad\left.+\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}-1\right]\\
&= (1+|B|)\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[1+\frac{3|ab|}{(c-|a|-|b|-1)}\right.\\
&\quad\left.+\frac{(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2}\right]-(1+|B|).
\end{aligned}$$

It is easy to see that the last expression is bounded above by $(B-A)(1-\alpha)\cos\lambda$ if (30) holds. □

Theorem 2.5. *Let $a,b\in\mathbb{C}^*$ and c be a real number such that $c>|a|+|b|+1$. If the inequality (23) satisfied, then $[I_{a,b,c}(f)](z)$ maps the class $\mathcal{K}$ to the class $R^{\lambda}(A,B,\alpha)$.*

Proof. It suffices to prove that

$$\sum_{n=2}^{\infty} n(1+|B|)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right||a_n| \le (B-A)(1-\alpha)\cos\lambda. \tag{32}$$

By inequality (25) and the fact that $f(z)\in\mathcal{K}$ (i.e., $|a_n|\le 1$ *if* $f(z)\in\mathcal{A}$) (see [1]), we obtain the required result. □

Theorem 2.6. *Let $a,b\in\mathbb{C}^*$ and c be a real number. If, for some k, the following inequality*

$${}_3F_2(|a|,|b|,P_1;c,1;1)\le\left[1+\frac{(B-A)(1-\alpha)\cos\lambda}{1+|B|}\right], \tag{33}$$

is true, then $[I_{a,b,c}(f)](z)$ maps the class $k-\mathcal{UCV}$ to the class $R^{\lambda}(A,B,\alpha)$.

Proof. We need to prove that

$$\sum_{n=2}^{\infty} n(1+|B|)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right| \le (B-A)(1-\alpha)\cos\lambda. \tag{34}$$

The left hand side of (34), by (25) and the sufficient condition (15), is less than or equal to

$$\sum_{n=2}^{\infty} n(1+|B|)\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}\frac{(P_1)_{n-1}}{n!}=T_5.$$

Now

$$\begin{aligned}
T_5 &= (1+|B|)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-1}}\\
&= (1+|B|)\left[{}_3F_2(|a|,|b|,P_1;c,1;1)-1\right].
\end{aligned}$$

From (34), we obtain the required result. □

Theorem 2.7. *Let $a,b\in\mathbb{C}^*$ and c be a real number. If, for some k, the following inequality*

$$\begin{aligned}
&\frac{|ab|P_1}{c}\,{}_3F_2(|a|+1,|b|+1,P_1+1;c+1,2;1)\\
&\quad+{}_3F_2(|a|,|b|,P_1;c,1;1)\\
&\quad\le\left[1+\frac{(B-A)(1-\alpha)\cos\lambda}{1+|B|}\right],
\end{aligned}\tag{35}$$

satisfied, then $[I_{a,b,c}(f)](z)$ maps the class $k-\mathcal{ST}$ to the class $R^{\lambda}(A,B,\alpha)$.

Proof. It is enough to show that

$$\sum_{n=2}^{\infty} n(1+|B|)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right| \le (B-A)(1-\alpha)\cos\lambda. \tag{36}$$

The left hand side of (36), by (25) and the sufficient condition (17), is less than or equal to

$$\sum_{n=2}^{\infty} n(1+|B|)\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}\frac{(P_1)_{n-1}}{(n-1)!}=T_6,$$

and

$$\begin{aligned}
T_6 &= (1+|B|)\left[\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-2}}\right.\\
&\quad\left.+\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}(P_1)_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-1}}\right]\\
&= (1+|B|)\left[\sum_{n=0}^{\infty}\frac{(|a|)_{n+1}(|b|)_{n+1}(P_1)_{n+1}}{(c)_{n+1}(1)_n(2)_n}\right.\\
&\quad\left.+{}_3F_2(|a|,|b|,P_1;c,1;1)-1\right]\\
&= (1+|B|)\frac{|ab|P_1}{c}\,{}_3F_2(|a|+1,|b|+1,P_1+1;c+1,2;1)\\
&\quad+(1+|B|)\left[{}_3F_2(|a|,|b|,P_1;c,1;1)-1\right].
\end{aligned}$$

Now, the proof of Theorem 2.7 is completed. □

Theorem 2.8. *Let $a, b \in \mathbb{C}^*$ and c be a real number such that $c > |a| + |b| + 1$. If, for some k, the following inequality*

$$(B-A)(1-\alpha)\cos\lambda \frac{|ab|\Gamma(c)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)} \leq \frac{1}{(k+2)}, \tag{37}$$

satisfied, then $[I_{a,b,c}(f)](z)$ maps the class $R^\lambda(A, B, \alpha)$ to the class $k-\mathcal{UCV}$.

Proof. By the sufficient condition (21), we need to show that

$$\sum_{n=2}^{\infty} n(n-1)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right| \leq \frac{1}{(k+2)}. \tag{38}$$

The left hand side of (38), by (25) and the sufficient condition (20), is less than or equal to

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(n-1)\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}\frac{(B-A)(1-\alpha)\cos\lambda}{n} \\
&\quad = (B-A)(1-\alpha)\cos\lambda \sum_{n=2}^{\infty}(n-1)\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\
&\quad = (B-A)(1-\alpha)\cos\lambda \frac{|ab|}{c}\frac{\Gamma(c+1)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)} \\
&\quad = (B-A)(1-\alpha)\cos\lambda \frac{|ab|\Gamma(c)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)}.
\end{aligned}$$

We note that the last expression is bounded above by $\frac{1}{(k+2)}$ if (37) holds. □

Theorem 2.9. *Let $a, b \in \mathbb{C}^*$ $(|a| \neq 1,\ |b| \neq 1)$ and c be a real number such that $c > \max\{0,\ |a|+|b|-1\}$. If, for some k, the following inequality*

$$\begin{aligned}
&\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[(k+1) - \frac{k(c-|a|-|b|)}{(|a|-1)(|b|-1)}\right] \\
&\quad \leq 1 + \frac{1}{(B-A)(1-\alpha)\cos\lambda} + \frac{k(1-c)}{(|a|-1)(|b|-1)}.
\end{aligned} \tag{39}$$

satisfied, then $[I_{a,b,c}(f)](z)$ maps the class $R^\lambda(A, B, \alpha)$ to the class $k-\mathcal{ST}$.

Proof. Making use of the sufficient condition (22), it is enough to prove that

$$\sum_{n=2}^{\infty}[n(1+k)-k]\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right| < 1. \tag{40}$$

The left hand side of (40), by (25) and the sufficient condition (20), is less than or equal to

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n(1+k)-k]\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}\frac{(B-A)(1-\alpha)\cos\lambda}{n} \\
&\quad = (B-A)(1-\alpha)(1+k)\cos\lambda \sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\
&\qquad - k(B-A)(1-\alpha)\cos\lambda \sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n}} \\
&\quad = (B-A)(1-\alpha)(1+k)\cos\lambda \sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\
&\qquad - k(B-A)(1-\alpha)\cos\lambda \frac{(c-1)}{(|a|-1)(|b|-1)} \\
&\qquad \times\left[\sum_{n=0}^{\infty}\frac{(|a|-1)_n(|b|-1)_n}{(c-1)_n(1)_n} - 1 - \frac{(|a|-1)(|b|-1)}{(c-1)}\right] \\
&\quad = (B-A)(1-\alpha)\cos\lambda \frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)} \\
&\qquad \times\left[(1+k) - k\frac{(c-|a|-|b|)}{(|a|-1)(|b|-1)}\right] \\
&\qquad - (B-A)(1-\alpha)\cos\lambda\left[1 - \frac{k(c-1)}{(|a|-1)(|b|-1)}\right].
\end{aligned}$$

By a simplification, we see that the last expression is bounded above by 1 if (39) holds. □

Remark. By specializing A, B and α in the above theorems, we will obtain new results for different classes mentioned in the introduction.

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